WEAK TYPE ESTIMATES ON CERTAIN HARDY SPACES FOR SMOOTH CONE TYPE MULTIPLIERS

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Abstract. Let $\varrho \in C^\infty(\mathbb{R}^d \setminus \{0\})$ be a non-radial homogeneous distance function satisfying $\varrho(t\xi) = t\varrho(\xi)$. For $f \in \mathfrak{S}(\mathbb{R}^{d+1})$ and $\delta > 0$, we consider convolution operator $\mathcal{T}^\delta$ associated with the smooth cone type multipliers defined by

$$\widehat{\mathcal{T}^\delta f}(\xi,\tau) = \left(1 - \frac{\varrho(\xi)}{|\tau|}\right)_+^\delta \hat{f}(\xi,\tau),\ (\xi,\tau) \in \mathbb{R}^d \times \mathbb{R}.$$

If the unit sphere $\Sigma_\varrho \risingdotseq \{\xi \in \mathbb{R}^d : \varrho(\xi) = 1\}$ is a convex hypersurface of finite type, then we prove that the operator $\mathcal{T}^{\delta(p)}$ maps from $H^p(\mathbb{R}^{d+1})$, $0 < p < 1$, into weak-$L^p(\mathbb{R}^{d+1})$ for the critical index $\delta(p) = d(1/p - 1/2) - 1/2$.

1. Introduction.

Let M be a real-valued $d \times d$ matrix whose eigenvalues have positive real part. Then the linear transformations $A_t = \exp(M \log t)$, $t > 0$, form a dilation group on $\mathbb{R}^d$. Let $\varrho \in C^\infty(\mathbb{R}^d \setminus \{0\})$ be a positive real-valued function satisfying $\varrho(A_t\,\xi) = t\,\varrho(\xi)$, which is called an A_t-homogeneous distance function (refer to [11] for elementary properties).

Let $\mathfrak{S}(\mathbb{R}^{d+1})$ be the Schwartz space of rapidly decreasing $C^\infty(\mathbb{R}^{d+1})$-functions, and let $\hat{f}$ be the Fourier transform of $f \in \mathfrak{S}(\mathbb{R}^{d+1})$. In what follows, we shall always assume that $A_t = t\,I$ and $\varrho \in C^\infty(\mathbb{R}^d \setminus \{0\})$ is a non-radial A_t-homogeneous distance function whose unit sphere

$$\Sigma_\varrho \risingdotseq \{\xi \in \mathbb{R}^d : \varrho(\xi) = 1\}$$

satisfies a finite type condition, i.e. every tangent line makes finite order of contact with Σ_ϱ. For $f \in \mathfrak{S}(\mathbb{R}^{d+1})$ and $\delta > 0$, we define convolution operators $\mathcal{T}^\delta$ by

$$\mathcal{T}^\delta f(x,t) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle + it\tau} \left(1 - \frac{\varrho(\xi)}{|\tau|}\right)_+^\delta \hat{f}(\xi,\tau)\, d\xi\, d\tau,\ (\xi,\tau) \in \mathbb{R}^d \times \mathbb{R}.$$

In the case that $A_t = t^{1/2}\, I$ and $\varrho(\xi) = |\xi|^2$, there is no optimal $L^p(\mathbb{R}^d)$-mapping properties of $\mathcal{T}^\delta$ which are known for $p > 1$ and $d \geq 2$. For partial results related with this, the reader can refer to G. Mockenhaupt [7] and J. Bourgain [1]. However, it is known (see [5]) about its sharp weak type results on $H^p(\mathbb{R}^{d+1})$, $0 < p < 1$, with the critical index $\delta(p) = d(1/p - 1/2) - 1/2$.

The purpose of this article is to obtain sharp weak type endpoint ($\delta(p) = d(1/p - 1/2) - 1/2$) results of $\mathcal{T}^\delta$ on $H^p(\mathbb{R}^{d+1})$, $0 < p < 1$, for the distance function $\varrho(\xi)$ in the sense that $\mathcal{T}^\delta$ is a bounded operator of $H^p(\mathbb{R}^{d+1})$ into $L^p(\mathbb{R}^{d+1})$ for $\delta > \delta(p)$, but it fails to be of restricted weak type (p,p) on $H^p(\mathbb{R}^{d+1})$ for all $\delta < \delta(p)$ and fails to be a bounded operator of $H^p(\mathbb{R}^{d+1})$ into $L^p(\mathbb{R}^{d+1})$ when $\delta = \delta(p)$. Here H^p denotes the standard real Hardy space as defined by E. M. Stein in [9].

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Theorem 1.1. *Suppose that $\varrho \in C^\infty(\mathbb{R}^d \setminus \{0\})$ is a non-radial homogeneous distance function satisfying $\varrho(t\xi) = t\varrho(\xi)$ whose unit sphere Σ_ϱ is a convex hypersurface of finite type. If $\delta(p) = d(1/p - 1/2) - 1/2$ for $0 < p < 1$, then $\mathcal{T}^{\delta(p)}$ maps $H^p(\mathbb{R}^{d+1})$ boundedly into weak-$L^p(\mathbb{R}^{d+1})$; that is, there exists a constant $C > 0$ not depending upon λ and f such that for any $f \in H^p(\mathbb{R}^{d+1})$,*

$$\left|\{(x,t) \in \mathbb{R}^{d+1} : |\mathcal{T}^{\delta(p)} f(x,t)| > \lambda\}\right| \le C\, \frac{\|f\|^p_{H^p(\mathbb{R}^{d+1})}}{\lambda^p},\ \lambda > 0.$$

Remarks. (i) This result generalizes that of [5] in $\mathbb{R}^d \times \mathbb{R}$ to non-radial cases (which is of finite type and convex) by using the arguments based on the results in [6].

(ii) As a matter of fact, we prove this result under more general surface condition than the finite type condition on Σ_ϱ, which was so called a spherically integrable condition of order < 1 in [6].

In what follows, we shall use the polar coordinates; given $x \in \mathbb{R}^d$, we write $x = r\,\theta$ where $r = |x|$ and $\theta = (\theta_1, \theta_2, \cdots, \theta_d) \in S^{d-1}$. Given two quantities A and B, we write $A \lesssim B$ or $B \gtrsim A$ if there is a positive constant c (possibly depending on the dimension $d \ge 2$, the hypersurface Σ_ϱ, and the index p to be given) such that $A \le cB$. We also write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. We denote by $\mathbb{R}^d_0 \fallingdotseq \mathbb{R}^d \setminus \{0\}$. For $e \in S^{d-1}$, we let $\mathcal{D}_e f$ denote the directional derivative $\langle e, \nabla_\xi \rangle f$ of a function $f(\xi)$ defined on $\mathbb{R}^d$. For $\mathfrak{e} \in S^{d-2}$, we let $\mathcal{D}^w_{\mathfrak{e}} g$ denote the directional derivative $\langle \mathfrak{e}, \nabla_w \rangle g$ of a function $g(w)$ defined on $\mathbb{R}^{d-1}$. For a multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{d+1}) \in (\mathbb{N} \cup \{0\})^{d+1}$, we denote its size $|\alpha|$ by $|\alpha| \fallingdotseq \alpha_1 + \alpha_2 + \cdots + \alpha_{d+1}$.

2. Preliminary estimates on a smooth convex hypersurface of finite type.

Let Σ be a smooth convex hypersurface of $\mathbb{R}^d$ and let $d\sigma$ be the induced surface area measure on Σ. For $x \in \mathbb{R}^d$, we denote by $\mathcal{B}(\xi(x), s)$ the spherical cap near $\xi(x) \in \Sigma$ cut off from Σ by a plane parallel to $T_{\xi(x)}(\Sigma)$ (the affine tangent plane to Σ at $\xi(x)$) at distance $s > 0$ from it; that is,

$$\mathcal{B}(\xi(x), s) = \{\xi \in \Sigma :\ d(\xi, T_{\xi(x)}(\Sigma)) < s\},$$

where $\xi(x)$ is the point of Σ whose outer unit normal is in the direction x. These spherical caps play an important role in furnishing the decay of the Fourier transform of the measure $d\sigma$. It is well known [9] that the function

$$\varPhi(\theta) \fallingdotseq \sup_{r>0} \sigma[\mathcal{B}(\xi(r\theta), 1/r)](1+r)^{\frac{d-1}{2}} \tag{2.1}$$

is bounded on S^{d-1} provided that Σ has nonvanishing Gaussian curvature.

Remark. (i) B. Randol [8] proved that if Σ is a real analytic convex hypersurface of $\mathbb{R}^d$ then $\varPhi \in L^p(S^{d-1})$ for some $p > 2$. Thus any real analytic convex hypersurface satisfies that $\varPhi \in L^p(S^{d-1})$ for any $p \le 2$.

(ii) More generally, it was shown by I. Svensson [12] that if Σ is a smooth convex hypersurface of finite type $k \ge 2$ then $\varPhi \in L^p(S^{d-1})$ for some $p > 2$.

Thus, by the above remark (ii), it is natural for us to obtain the following corollary.

Corollary 2.1. *If Σ is a smooth convex hypersurface of $\mathbb{R}^d$ which is of finite type, then $\varPhi \in L^p(S^{d-1})$ for any $p \le 2$.*

Sharp decay estimates for the Fourier transform of surface measure on a smooth convex hypersurface Σ of finite type $k \ge 2$ have been obtained by Bruna, Nagel, and Wainger [2]; precisely speaking,

$$|\mathcal{F}[d\sigma](x)| \sim \sigma[\mathcal{B}(\xi(x), 1/|x|)]. \tag{2.2}$$

They define a family of anisotropic balls on Σ by letting

$$\mathcal{B}(\xi_0, s) = \{\xi \in \Sigma : \, d(\xi, T_{\xi_0}(\Sigma)) < s\}$$

where $\xi_0 \in \Sigma$. We now recall some properties of the anisotropic balls $\mathcal{B}(\xi_0, s)$ associated with Σ. The proof of the doubling property in [2] makes it possible to obtain the following stronger estimate for the surface measure of these balls;

$$\sigma[\mathcal{B}(\xi_0, \gamma\, s)] \lesssim \begin{cases} \gamma^{\frac{d-1}{2}} \sigma[\mathcal{B}(\xi_0, s)], & \gamma \geq 1, \\ \gamma^{\frac{d-1}{k}} \sigma[\mathcal{B}(\xi_0, s)], & \gamma < 1. \end{cases}$$

It also follows from the triangle inequality and the doubling property [2] that there is a positive constant $C > 0$ independent of $s > 0$ such that

$$\frac{1}{C}\, \sigma[\mathcal{B}(\xi_0, s)] \leq \sigma[\mathcal{B}(\xi, s)] \leq C\, \sigma[\mathcal{B}(\xi_0, s)] \quad \text{for any } \xi \in \mathcal{B}(\xi_0, s). \tag{2.3}$$

Lemma 2.2 [6]. *Let Σ be a smooth convex hypersurface of $\mathbb{R}^d$ which is of finite type $k \geq 2$. Then there is a constant $C = C(\Sigma) > 0$ such that for any $y \in B(0; s)$ and $x \in B(0; 2s)^c$, $0 < s \leq 1$,*

$$\xi(x - y) \in \mathcal{B}(\xi(x), C/|x|)$$

where $\xi(x)$ is the point of Σ whose outer unit normal is in the direction x.

Lemma 2.3. *Let Σ be a smooth convex hypersurface of $\mathbb{R}^d$ which is of finite type $k \geq 2$. Then there is a constant $C = C(\Sigma) > 0$ such that for any $x, y \in \mathbb{R}^d$ with $|x| > 2\,|y| > 0$,*

$$\Phi\left(\frac{x-y}{|x-y|}\right) \leq C\ \Phi\left(\frac{x}{|x|}\right)$$

where Φ is the function defined as in (2.1).

Proof. It easily follows from (2.3), the definition of Φ, and Lemma 2.2 that for any $y \in B(0; s)$ and $x \in B(0; 2s)^c$, $0 < s \leq 1$,

$$\begin{aligned} \Phi\left(\frac{x-y}{|x-y|}\right) &\lesssim \sup_{r>0} \sigma[\mathcal{B}(\xi(x-y), 1/r)]\, (1+r)^{\frac{d-1}{2}} \\ &\lesssim \sup_{r>0} \sigma[\mathcal{B}(\xi(x), 1/r)]\, (1+r)^{\frac{d-1}{2}} = \Phi\left(\frac{x}{|x|}\right). \end{aligned}$$

Thus this implies that for any $x, y \in \mathbb{R}^d$ with $|x| > 2\,|y| > 0$,

$$\Phi\left(\frac{x-y}{|x-y|}\right) = \Phi\left(\frac{x/|y| - y/|y|}{|x/|y| - y/|y||}\right) \lesssim \Phi\left(\frac{x/|y|}{|x/|y||}\right) = \Phi\left(\frac{x}{|x|}\right). \qquad \square$$

3. The kernel estimate associated with the cone type multipliers.

Throughout this section from now on, we shall concentrate upon obtaining decay estimate of the kernel associated with the cone type multipliers. First of all, we obtain the lower bound of the phase function of the kernel on the dual cone $\Gamma_\gamma = \{(x,t) \in \mathbb{R}^d \times \mathbb{R} : |t| \geq \gamma|x|\}, \gamma = \sup_{\xi \in \mathcal{B}_\varrho} |\xi|$, and the unit ball $\mathcal{B}_\varrho \fallingdotseq \{\xi \in \mathbb{R}^d : \varrho(\xi) \leq 1\}$.

Lemma 3.1. *Suppose that $\varrho \in C^\infty(\mathbb{R}^d \setminus \{0\})$ is a non-radial homogeneous distance function satisfying $\varrho(t\xi) = t\varrho(\xi)$. Then we have the following estimate*

$$\inf_{\xi \in \mathcal{B}_\varrho} |t + \langle x, \xi\rangle| \geq |t| - \gamma|x| \geq 0, \ \ (x,t) \in \Gamma_\gamma.$$

Proof. It easily follows from the triangle inequality and Schwarz inequality. $\square$

Let $\psi \in C_0^\infty(\mathbb{R})$ be supported in $(1/2, 2)$ such that $\sum_{l \in \mathbb{Z}} \psi(2^{-l}\, t) = 1$ for $t > 0$. For fixed $l \in \mathbb{Z}$, we shall need pointwise estimates for the operators

$$\mathcal{T}_l^{\delta(p)} f(x,t) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathcal{K}_l^{\delta(p)}(x-y, t-s)\, f(y,s)\, dy\, ds$$

where

$$\mathcal{K}_l^{\delta(p)}(x,t) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle + it\tau} \left(1 - \frac{\varrho(\xi)}{|\tau|}\right)_+^{\delta(p)} \psi(2^{-l}\, \tau)\, d\xi\, d\tau. \tag{3.1}$$

For each $l \in \mathbb{Z}$, the kernel $\mathcal{K}_l^{\delta(p)}$ has the property

$$\mathcal{K}_l^{\delta(p)}(x,t) = 2^{(d+1)l}\, \mathcal{K}_0^{\delta(p)}(2^l x, 2^l t). \tag{3.2}$$

Let $\phi \in C_0^\infty(\mathbb{R})$ be supported in $(1/2, 2)$ such that $\sum_{k \in \mathbb{Z}} \phi(2^k\, s) = 1$ for $0 < s < 1$, and let

$$\mathcal{K}_{k,l}^{\delta(p)}(x,t) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle + it\tau} \phi\left(2^{k+1}\left(1 - \frac{\varrho(\xi)}{|\tau|}\right)\right) \left(1 - \frac{\varrho(\xi)}{|\tau|}\right)_+^{\delta(p)} \psi(2^{-l}\, \tau)\, d\xi\, d\tau.$$

We write $\sum_{k \in \mathbb{N}} \mathcal{K}_{k,l}^{\delta(p)} + \mathcal{K}_{0,l}^{\delta(p)} = \mathcal{K}_l^{\delta(p)}$ and $\sum_{k \in \mathbb{N}} \mathcal{T}_{k,l}^{\delta(p)} + \mathcal{T}_{0,l}^{\delta(p)} = \mathcal{T}_l^{\delta(p)}$. Since the kernel $\mathcal{K}_{0,l}^{\delta(p)}(x,t)$ has a very nice decay in terms of x-variable and a properly good decay in t-variable, it suffices to treat only the operators $\mathcal{T}_{k,l}^{\delta(p)}$ in the following lemma.

Lemma 3.2. *Suppose that $\varrho \in C^\infty(\mathbb{R}^d \setminus \{0\})$ is a non-radial homogeneous distance function satisfying $\varrho(t\xi) = t\varrho(\xi)$. If $|x| \leq 2^{2-l}\gamma^{-1}$ for fixed $l \in \mathbb{Z}$, then for each $k \in \mathbb{N}$ we have the following uniform estimates; for any $N \in \mathbb{N}$,*

$$|\mathcal{K}_{k,l}^{\delta(p)}(x,t)| + \sum_{|\alpha| = n+1} \frac{1}{\alpha!} |[\mathcal{D}^\alpha \mathcal{K}_{k,l}^{\delta(p)}](x,t)| \leq C\, 2^{(d+1)l} 2^{-k(\delta(p)+1)} \min\{1, (2^{-k} 2^l\, |t|)^{-N}\},$$

and thus

$$|\mathcal{K}_l^{\delta(p)}(x,t)| + \sum_{|\alpha| = n+1} \frac{1}{\alpha!} |[\mathcal{D}^\alpha \mathcal{K}_l^{\delta(p)}](x,t)| \leq C\, 2^{(d+1)l} \frac{1}{(1 + 2^l\, |t|)^{\delta(p)+1}}$$

where $n \in \mathbb{N}$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{d+1}) \in (\mathbb{N} \cup \{0\})^{d+1}$ is a multi-index, and we denote by $\alpha! = \alpha_1!\, \alpha_2! \cdots \alpha_{d+1}!$ and

$$\mathcal{D}^\alpha \mathcal{K}_{k,l}^{\delta(p)}(x,t) = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} \left(\frac{\partial}{\partial t}\right)^{\alpha_{d+1}} \mathcal{K}_{k,l}(x,t).$$

Proof. It easily follows from the integration by parts N-times with respect to τ. $\square$

Now we proceed the case that $|x| > 2^{2-l}\gamma^{-1}$. By the change of variables, the integral (3.1) becomes

$$\mathcal{K}_l^{\delta(p)}(x,t) = 2^{(d+1)l}\int_{\mathbb{R}}\int_{\mathbb{R}^d} e^{i2^l|x|\langle\frac{x}{|x|},\xi\rangle\tau + i2^l t\tau}(1-\varrho(\xi))_+^{\delta(p)}\,\psi(\tau)\,\tau^2\,d\xi\,d\tau.$$

We shall employ a decomposition of the Bochner-Riesz multiplier $(1-\varrho)_+^{\delta(p)}$ as in A. Córdoba [3]. Let $\phi \in C_0^\infty(1/2,2)$ satisfy $\sum_{k\in\mathbb{Z}}\phi(2^k t) = 1$ for $t>0$. Put $\Phi_k^{\delta(p)} = \phi(2^{k+1}(1-\varrho))(1-\varrho)_+^{\delta(p)}$ and $\Phi_0^{\delta(p)} = (1-\varrho)_+^{\delta(p)} - \sum_{k\in\mathbb{N}}\Phi_k^{\delta(p)}$ for $k\in\mathbb{N}$. Then we note that $\sum_{k\in\mathbb{N}}\Phi_k^{\delta(p)} = \varphi\,(1-\varrho)_+^{\delta(p)}$ *a.e.*, where $\varphi\in C_0^\infty(\mathbb{R}^d)$ is a function supported in the closed annulus

$$\{\xi\in\mathbb{R}^d : 1/2 \le \varrho(\xi) \le 2\}$$

such that

$$\varphi(\xi) = \sum_{k\in\mathbb{N}}\phi(2^{k+1}(1-\varrho(\xi))) \text{ on the open annulus } \{\xi\in\mathbb{R}^d : 1/2<\varrho(\xi)<1\}.$$

We now introduce a partition of unity Ξ_ℓ, $\ell = 1,2,\cdots,N_0$, on the unit sphere Σ_ϱ which we extend to $\mathbb{R}^d$ by way of $\Pi_\ell(t\,\zeta) = \Xi_\ell(\zeta)$, $t>0$, $\zeta\in\Sigma_\varrho$, and which satisfies the following properties; by compactness of Σ_ϱ, there are a sufficiently large finite number of points $\zeta_1,\zeta_2,\cdots,\zeta_{N_0}\in\Sigma_\varrho$ such that for $\ell=1,2,\cdots,N_0$,

(i) $\sum_{\ell=1}^{N_0}\Pi_\ell(\zeta)\equiv 1$ for all $\zeta\in\Sigma_\varrho$,

(ii) $\Xi_\ell(\zeta) = 1$ for all $\zeta\in\Sigma_\varrho\cap B(\zeta_\ell;2^{-M/2})$,

(iii) Ξ_ℓ is supported in $\Sigma_\varrho\cap B(\zeta_\ell;2^{1-M/2})$,

(iv) $|\mathcal{D}^\alpha\Pi_\ell(\xi)|\lesssim 2^{|\alpha|M/2}$ for any multi-index α, if $1/2\le\varrho(\xi)\le 2$,

(v) $N_0\lesssim 2^{(n-1)M/2}$ for some sufficiently large fixed M (to be chosen later),

where $B(\zeta_0;s)$ denotes the ball in $\mathbb{R}^d$ with center $\zeta_0\in\Sigma_\varrho$ and radius $s>0$. Then we split the kernel $\mathcal{K}_l^{\delta(p)}(x,t)$ into finitely many pieces as follows;

$$\begin{aligned}\mathcal{K}_l^{\delta(p)}&(x,t)\\ &= 2^{(d+1)l}\sum_{\ell=1}^{N_0}\iint e^{i2^l|x|\langle\frac{x}{|x|},\xi\rangle\tau+i2^l t\tau}(1-\varrho(\xi))_+^{\delta(p)}\varphi(\xi)\,\Pi_\ell(\xi)\,\psi(\tau)\,\tau^d\,d\xi\,d\tau\\ &\fallingdotseq \sum_{\ell=1}^{N_0}\mathcal{K}_{l\ell}^{\delta(p)}(x,t).\end{aligned}\tag{3.3}$$

Here we observe that for $l\in\mathbb{Z}$ and $\ell = 1,2,\cdots,N_0$,

$$\mathcal{K}_{l\ell}^{\delta(p)}(x,t) = 2^{(d+1)l}\mathcal{K}_{0\ell}^{\delta(p)}(2^l x, 2^l t).\tag{3.4}$$

Lemma 3.3. *Suppose that $\varrho\in C^\infty(\mathbb{R}^d\setminus\{0\})$ is a non-radial homogeneous distance function satisfying $\varrho(t\xi)=t\varrho(\xi)$. If $\delta(p) = d(1/p-1/2)-1/2$ for $0<p<1$, then given $n,N\in\mathbb{N}$, there exists a constant $C = C(p,n,N,\Sigma_\varrho)$ such that for any $(x,t)\in\Gamma_\gamma$,*

$$\begin{aligned}&|\mathcal{K}_{0\ell}^{\delta(p)}(x,t)| + \sum_{|\alpha|=n+1}\frac{1}{\alpha!}\,|[\mathcal{D}^\alpha\mathcal{K}_{0\ell}^{\delta(p)}](x,t)|\\ &\le \frac{C}{(1+|x|)^{\delta(p)+1+\frac{d-1}{2}}}\,\Phi\left(\frac{x}{|x|}\right)\frac{1}{(1+|t|-\gamma|x|)^N}\end{aligned}$$

where $\alpha \in (\mathbb{N} \cup \{0\})^d$ is a multi-index satisfying $|\alpha| = n+1$.

Proof. We observe that the map

$$\mathbb{R}_+ \times \Sigma_\varrho \to \mathbb{R}^d_0,\ (\varrho, \zeta) \mapsto \varrho\,\zeta = \xi,\ \zeta \in \Sigma_\varrho$$

defines polar coordinates with respect to ϱ by way of

$$d\xi = \varrho^{d-1} \langle \zeta, \mathfrak{n}(\zeta) \rangle \, d\varrho \, d\sigma(\zeta),$$

where $d\sigma(\zeta)$ denotes the surface area measure on Σ_ϱ and $\mathfrak{n}(\zeta)$ is the outer unit normal vector to Σ_ϱ at $\zeta \in \Sigma_\varrho$. Now fix $\zeta_\ell \in \Sigma_\varrho$. Then the unit sphere Σ_ϱ can be parametrized near $\zeta_\ell \in \Sigma_\varrho$ by a map

$$w \mapsto \mathcal{P}(w),\ w \in \mathbb{R}^{d-1},\ |w| < 1$$

such that $\mathcal{P}(0) = \zeta_\ell$. Then there is a neighborhood $\mathcal{U}_0$ of ζ_ℓ with compact closure, and a neighborhood $\mathcal{V}_0$ of the origin in $\mathbb{R}^{d-1}$ so that the map

$$\mathcal{Q} : (1/2, 3/2) \times \mathcal{V}_0 \to \mathcal{U}_0,\ (\varrho, w) \mapsto \mathcal{Q}(\varrho, w) = \varrho\,\mathcal{P}(w) \tag{3.5}$$

is a diffeomorphism with $\mathcal{Q}(1,0) = \zeta_\ell$. The Jacobian of $\mathcal{Q}$ is given by

$$\mathfrak{J}(\varrho, w) = \varrho^{d-1} \langle \mathcal{P}(w), \mathfrak{n}(\mathcal{P}(w)) \rangle \, \mathcal{R}(w),$$

where $\mathcal{R}(w)$ is positive and

$$[\mathcal{R}(w)]^2 = \det\left(\left[\frac{d\mathcal{P}}{dw}\right]^* \left[\frac{d\mathcal{P}}{dw}\right] \right).$$

By (3.3), we have that

$$\mathcal{K}^{\delta(p)}_{0\ell}(x,t) = \iint e^{i|x|\langle \frac{x}{|x|}, \xi\rangle \tau + it\tau} (1-\varrho(\xi))_+^{\delta(p)} \varphi(\xi)\, \Pi_\ell(\xi)\, \tau^d\, \psi(\tau)\, d\xi\, d\tau. \tag{3.6}$$

Let $\mathcal{I}^0_\ell(x,\tau)$ denote the integral with respect to ξ-variable in (3.6); that is to say,

$$\mathcal{I}^0_\ell(x,\tau) = \int_{\mathbb{R}^d} e^{i|x|\langle \frac{x}{|x|}, \xi\rangle \tau} (1-\varrho(\xi))_+^{\delta(p)} \varphi(\xi)\, \Pi_\ell(\xi)\, d\xi.$$

Applying generalized polar coordinates that we introduced in the above, we have that

$$\begin{aligned} &\mathcal{I}^0_\ell(x,\tau) \\ &= \iint e^{i|x|\langle \frac{x}{|x|}, \varrho\,\zeta\rangle \tau} (1-\varrho)_+^{\delta(p)}\, \varphi(\varrho\,\zeta)\, \Pi_\ell(\varrho\,\zeta)\, \varrho^{d-1} \langle \zeta, \mathfrak{n}(\zeta)\rangle\, d\varrho\, d\sigma(\zeta) \\ &= \int \left[\int e^{i|x|\langle \frac{x}{|x|}, \varrho\,\mathcal{P}(w)\rangle \tau}\, \varphi[\mathcal{Q}(\varrho,w)]\, \Pi_\ell[\mathcal{Q}(\varrho,w)]\, \mathfrak{J}(\varrho,w)\, dw \right] (1-\varrho)_+^{\delta(p)}\, d\varrho \\ &\eqqcolon \int \mathcal{J}^0_\ell(x,\tau,\varrho)\, (1-\varrho)_+^{\delta(p)}\, d\varrho. \end{aligned} \tag{3.7}$$

We note that if $\langle \theta, \mathfrak{n}(\zeta_\ell)\rangle < 1$, then we have that

$$\nabla_w \langle \theta, \mathcal{P}(w) \rangle \,|_{w=0} \neq 0.$$

Combining this with the homogeneity condition on the distance function ϱ and choosing a sufficiently large $M > 0$ in (i) $\sim$ (v), we may select $\varepsilon_0 > 0$, a neighborhood $\mathcal{U}_1$ of ζ_ℓ with supp $(\varphi\,\Pi_\ell) \subset \overline{\mathcal{U}_1} \subset \mathcal{U}_0$, and a neighborhood $\mathcal{V}_1$ of the origin in $\mathbb{R}^{d-1}$ so that (3.5) satisfies, and such that for all $(w,\varrho) \in \overline{\mathcal{V}_1} \times [1/2,1]$,

$$|\nabla_w \langle \theta, \varrho\,\mathcal{P}(w)\rangle| \geq c_0 \quad \text{if } |\langle \theta, \mathfrak{n}(\zeta_\ell)\rangle| \leq 1-\varepsilon_0$$

and

$$c_1 \leq \left| \frac{\partial}{\partial \varrho} \langle \theta, \varrho\,\mathcal{P}(w)\rangle \right| \leq c_2 \quad \text{if } |\langle \theta, \mathfrak{n}(\zeta_\ell)\rangle| \geq 1-\varepsilon_0 \tag{3.8}$$

for some $c_0 > 0$, $c_1 > 0$, and $c_2 > 0$. We choose some $\mathfrak{e} \in S^{d-2}$ so that for all $(w,\varrho) \in \overline{\mathcal{V}_1} \times [1/2,1]$,

$$|\mathcal{D}^w_{\mathfrak{e}} \langle \theta, \varrho\,\mathcal{P}(w)\rangle| \geq \frac{1}{2}\,|\nabla_w \langle \theta, \varrho\,\mathcal{P}(w)\rangle| \geq \frac{1}{2}\,c_0 \quad \text{if } |\langle \theta, \mathfrak{n}(\zeta_\ell)\rangle| \leq 1-\varepsilon_0. \tag{3.9}$$

If $|\langle \theta, \mathfrak{n}(\zeta_\ell)\rangle| \leq 1-\varepsilon_0$, we apply the integration of $\mathcal{J}^0_\ell(x,\tau,\varrho)$ by parts with respect to w-variable N_1-times to obtain that

$$\mathcal{J}^0_\ell(x,\tau,\varrho) = \int e^{i|x|\langle \frac{x}{|x|}, \varrho\,\mathcal{P}(w)\rangle \tau}\, \frac{(\mathcal{D}^w_{\mathfrak{e}})^{N_1}\, (\varphi[\mathcal{Q}(\varrho,w)]\,\Pi_\ell[\mathcal{Q}(\varrho,w)]\,\mathfrak{J}(\varrho,w))}{(i\,|x|\,\mathcal{D}^w_{\mathfrak{e}}\langle \theta, \varrho\,\mathcal{P}(w)\rangle\,\tau)^{N_1}}\, dw.$$

Then we integrate the kernel $\mathcal{K}^{\delta(p)}_{0\ell}(x,t)$ by parts with respect to τ-variable N-times to get that

$$\begin{aligned} \mathcal{K}^{\delta(p)}_{0\ell}(x,t) = \iiint & e^{i|x|\langle \frac{x}{|x|}, \varrho\,\mathcal{P}(w)\rangle \tau + it\tau}\, \frac{(\frac{\partial}{\partial\tau})^N \{\tau^{d-N_1}\,\psi(\tau)\}}{[i(t+|x|\langle \frac{x}{|x|}, \varrho\,\mathcal{P}(w)\rangle)]^N}\, d\tau \\ & \times \frac{(\mathcal{D}^w_{\mathfrak{e}})^{N_1}\, (\varphi[\mathcal{Q}(\varrho,w)]\,\Pi_\ell[\mathcal{Q}(\varrho,w)]\,\mathfrak{J}(\varrho,w))}{(i\,|x|\,\mathcal{D}^w_{\mathfrak{e}}\langle \theta, \varrho\,\mathcal{P}(w)\rangle)^{N_1}}\, dw\,(1-\varrho)^{\delta(p)}_+\, d\varrho. \end{aligned} \tag{3.10}$$

Thus, by Lemma 3.1, (3.9), and (3.10), we have that for any $N, N_1 \in \mathbb{N}$,

$$|\mathcal{K}^{\delta(p)}_{0\ell}(x,t)| \leq \frac{C_{N_1}}{(1+|x|)^{N_1}}\, \frac{1}{(1+|t|-\gamma|x|)^N}. \tag{3.11}$$

If $|\langle \theta, \mathfrak{n}(\zeta_\ell)\rangle| \geq 1-\varepsilon_0$, then we apply the integration of $\mathcal{K}^{\delta(p)}_{0\ell}(x,t)$ (in (3.6) and (3.7)) by parts with respect to τ-variable N-times to obtain that

$$\begin{aligned} \mathcal{K}^{\delta(p)}_{0\ell}(x,t) &= \iiint e^{i|x|\langle \frac{x}{|x|}, \varrho\,\mathcal{P}(w)\rangle \tau + it\tau}\, \frac{(\frac{\partial}{\partial\tau})^N \{\tau^d\,\psi(\tau)\}}{[i(t+|x|\langle \frac{x}{|x|}, \varrho\,\mathcal{P}(w)\rangle)]^N}\, d\tau \\ &\quad \times \varphi[\mathcal{Q}(\varrho,w)]\,\Pi_\ell[\mathcal{Q}(\varrho,w)]\,\mathfrak{J}(\varrho,w)\, dw\,(1-\varrho)^{\delta(p)}_+\, d\varrho \\ &= \iiint e^{i|x|\langle \frac{x}{|x|}, \varrho\,\mathcal{P}(w)\rangle \tau}\,(1-\varrho)^{\delta(p)}_+ \\ &\quad \times \frac{\varphi[\mathcal{Q}(\varrho,w)]\,\Pi_\ell[\mathcal{Q}(\varrho,w)]\mathfrak{J}(\varrho,w)}{[i(t+|x|\langle \frac{x}{|x|}, \varrho\,\mathcal{P}(w)\rangle)]^N}\, d\varrho\, dw\;\; e^{it\tau} \left(\frac{\partial}{\partial\tau}\right)^N \{\tau^d \psi(\tau)\}\, d\tau. \end{aligned} \tag{3.12}$$

Then it follows from the asymptotic result (see [4]) of (3.12) with respect to ϱ-variable that for any $N_1 \in \mathbb{N}$,

$$\mathcal{K}^{\delta(p)}_{0\ell}(x,t) = \sum_{j=0}^{N_1-1} \int e^{it\tau}\, \mathfrak{h}_j(x,\tau) \left(\frac{\partial}{\partial\tau}\right)^N \{\tau^d \psi(\tau)\}\, d\tau + \mathcal{O}(|x|^{-N_1}), \tag{3.13}$$

where

$$\mathfrak{h}_j(x,\tau) = |x|^{-\delta(p)-1-j} \int_{\Sigma_\varrho} e^{i|x|\langle \frac{x}{|x|},\zeta\rangle\tau}\, \mathfrak{k}_j(\zeta, x/|x|)\, d\sigma(\zeta)$$

and $\mathfrak{k}_j \in C_0^\infty(\mathcal{P}(\mathcal{V}_0)\times S^{d-1})$ for $j = 0,1,2,\cdots,N_1-1$. In particular,

$$\mathfrak{k}_0(\zeta, x/|x|) = \Gamma(\delta(p)+1)\, e^{-i\pi(\delta(p)+1)/2}\, \frac{\Pi_\ell(\zeta)}{[i(t+|x|\langle \frac{x}{|x|},\zeta\rangle)]^N}\, \langle \zeta, \mathfrak{n}(\zeta)\rangle\, [\langle x/|x|, \zeta\rangle]^{-\delta(p)-1}.$$

If we restrict to $(x,t)\in\Gamma_\gamma$, then the required decay estimate of the kernel follows from (2.1), (2.2), Lemma 3.1, (3.8), (3.11), and (3.13). Finally, we can complete the proof by noting the fact that $\sum_{|\alpha|=N}\mathcal{D}^\alpha\mathcal{K}_{0\ell}^{\delta(p)} = \sum_{|\alpha|=N}(\mathcal{D}^\alpha\Psi)*\mathcal{K}_{0\ell}^{\delta(p)}$ for some Schwartz function $\Psi\in\mathfrak{S}(\mathbb{R}^{d+1})$. □

Corollary 3.4. *Suppose that $\varrho \in C^\infty(\mathbb{R}^d\setminus\{0\})$ is a non-radial homogeneous distance function satisfying $\varrho(t\xi) = t\varrho(\xi)$. If $\delta(p) = d(1/p-1/2)-1/2$ for $0<p<1$, then given $N\in\mathbb{N}$, there exists a constant $C = C(p,N,\Sigma_\varrho)$ such that for any $(x,t)\in\Gamma_\gamma$,*

$$\begin{aligned}|\mathcal{K}_0^{\delta(p)}(x,t)| &+ \sum_{|\alpha|=N}\frac{1}{\alpha!}\,|[\mathcal{D}^\alpha\mathcal{K}_0^{\delta(p)}](x,t)|\\ &\le \frac{C}{(1+|x|)^{\delta(p)+1+\frac{d-1}{2}}}\,\Phi\left(\frac{x}{|x|}\right)\frac{1}{(1+|t|-\gamma|x|)^N}\end{aligned}$$

where $\alpha\in(\mathbb{N}\cup\{0\})^d$ is a multi-index satisfying $|\alpha| = N$.

4. The atomic decomposition for H^p spaces and technical lemmas.

An atom is defined as follws: Let $0<p\le 1$ and ν be an integer satisfying $\nu\ge(d+1)(1/p-1)$. A (p,ν)-atom is a function $\mathfrak{a}$ which is supported on a cube Q with center $x_0\in\mathbb{R}^{d+1}$ and which satisfies

$$\text{(i) } |\mathfrak{a}(x)| \le |Q|^{-1/p} \quad\text{and}\quad \text{(ii) } \int_{\mathbb{R}^{d+1}} \mathfrak{a}(x)\, x^\alpha\, dx = 0,$$

where $\alpha = (\alpha_1,\alpha_2,\cdots,\alpha_{d+1})\in(\mathbb{N}\cup\{0\})^{d+1}$ is a multi-index satisfying $|\alpha| \doteqdot \alpha_1+\alpha_2+\cdots+\alpha_{d+1}\le\nu$ and $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_{d+1}^{\alpha_{d+1}}$. If $f = \sum_{i=1}^\infty c_i\mathfrak{a}_i$ where each $\mathfrak{a}_i$ is a (p,ν)-atom and $\{c_i\}\in\ell^p(\mathbb{R})$, then $f\in H^p$ and $\|f\|_{H^p}\lesssim\sum_i|c_i|^p$, and the converse inequality also holds (see [9]).

First of all, we recall a useful lemma [10] due to Stein, Taibleson and Weiss on summing up weak type functions in case of $0<p<1$.

Lemma 4.1. *Let $0<p<1$. Suppose that $\{\mathfrak{h}_k\}$ is a sequence of nonnegative measurable functions defined on a subset Ω of $\mathbb{R}^d$ such that*

$$|\{x\in\Omega:\, \mathfrak{h}_k(x)>\lambda\}| \le \frac{A}{\lambda^p},\ \lambda>0,$$

where $A>0$ is a constant. If $\{a_k\}$ is a sequence of positive numbers with $\|\{a_k\}\|_{\ell^p}<\infty$, then we have that

$$\left|\{x\in\Omega:\, \sum_k a_k\mathfrak{h}_k(x)>\lambda\}\right| \le \frac{2-p}{1-p}\, \|\{a_k\}\|_{\ell^p}^p\, \frac{A}{\lambda^p},\ \lambda>0.$$

Theorem 1.1 is obtained by applying a natural variant of Lemma 4.1 as in the following lemma.

Lemma 4.2. *Let $a > 0$ and $0 < p < 1$. Suppose that $\{\mathfrak{g}_l\}$ is a sequence of measurable functions defined on a subset Ω of $\mathbb{R}^d$ such that*

$$|\{x \in \Omega : |\mathfrak{g}_l(x)| > \lambda\}| \lesssim 2^{-alp}\,\lambda^{-p}$$

for $l \in \mathbb{N}$ and all $\lambda > 0$. Then we have that

$$\left|\{x \in \Omega : \sum_{l\in\mathbb{N}} |\mathfrak{g}_l(x)| > \lambda\}\right| \lesssim \lambda^{-p}.$$

For $j \in \mathbb{N}$ and $l \in \mathbb{Z}$, we set

$$\begin{aligned} A_l &= \{(x,t) \in \mathbb{R}^d \times \mathbb{R} : 2^l\gamma|x| \le 4, 2^l|t| > 2\},\\ B_l &= \{(x,t) \in \mathbb{R}^d \times \mathbb{R} : 2^l\gamma|x| > 4,\ 2^l\,|t| \le 2,\ ||t| - \gamma|x|| > 2^{-l}\},\\ C_l &= \{(x,t) \in \mathbb{R}^d \times \mathbb{R} : 2^l\gamma|x| > 4,\ 2^l\,|t| > 2,\ ||t| - \gamma|x|| \le 2^{-l}\},\\ D_l &= \{(x,t) \in \mathbb{R}^d \times \mathbb{R} : 2^l\gamma|x| > 4,\ 2^l\,|t| > 2,\ ||t| - \gamma|x|| > 2^{-l}\}\\ &\quad \bigcap \left(\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : |t| \le 2^{-1}\gamma\,|x|\} \cup \{(x,t) \in \mathbb{R}^d \times \mathbb{R} : |t| > 2\gamma|x|\}\right),\\ E_{jl} &= \{(x,t) \in \mathbb{R}^d \times \mathbb{R} : 2^l\gamma|x| > 4,\ 2^l\,|t| > 2,\ 2^{-l}2^{j-1} < ||t| - \gamma|x|| \le 2^{-l}2^j\}\\ &\quad \bigcap \{(x,t) \in \mathbb{R}^d \times \mathbb{R} : 2^{-1}\gamma\,|x| < |t| \le 2\gamma|x|\}. \end{aligned} \tag{4.1}$$

For $j \in \mathbb{N}$, $l \in \mathbb{Z}$, $(x,t) \in \mathbb{R}^d \times \mathbb{R}$, and $a, b, c \in \mathbb{R}_+$, we set

$$\begin{aligned} \mathcal{A}_l^c(x,t) &= 2^{lc}\,|t|^{-\delta(p)-1}\,\chi_{A_l}(x,t),\\ \mathcal{B}_l^b(x,t) &= 2^{lb}\,|x|^{-d/p-N}\ \Phi\left(\frac{x}{|x|}\right)\ \chi_{B_l}(x,t),\\ \mathcal{C}_l^a(x,t) &= 2^{la}\,|x|^{-d/p}\ \Phi\left(\frac{x}{|x|}\right)\ \chi_{C_l}(x,t),\\ \mathcal{D}_l^b(x,t) &= 2^{lb}\,|x|^{-d/p}\ \Phi\left(\frac{x}{|x|}\right)\ |t|^{-N}\ \chi_{D_l}(x,t),\\ \mathcal{E}_{jl}^a(x,t) &= 2^{la}2^{-jN}\,|x|^{-d/p}\ \Phi\left(\frac{x}{|x|}\right)\ \chi_{E_{jl}}(x,t). \end{aligned} \tag{4.2}$$

Then by simple computation we obtain the following lemma.

Lemma 4.3. *Let $0 < p < 1$ and $\delta(p) = d(1/p - 1/2) - 1/2$. Then we have the following estimates;*

(a) $\left|\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : \mathcal{A}_l^c(x,t) > \lambda\}\right| \lesssim 2^{lh}\,\lambda^{-p},\ \lambda > 0,$

(b) $\left|\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : \mathcal{B}_l^b(x,t) > \lambda\}\right| \lesssim 2^{lh}\,\lambda^{-p},\ \lambda > 0,$

(c) $\left|\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : \mathcal{C}_l^a(x,t) > \lambda\}\right| \lesssim 2^{lh}\,\lambda^{-p},\ \lambda > 0,$

(d) $\left|\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : \mathcal{D}_l^b(x,t) > \lambda\}\right| \lesssim 2^{lh}\,\lambda^{-p},\ \lambda > 0,$

(e) $\left|\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : \mathcal{E}_{jl}^a(x,t) > \lambda\}\right| \lesssim 2^{-j(Np-1)}\,2^{lh}\,\lambda^{-p},\ \lambda > 0,$ *where*

(i) $h = (d+1)(p-1)$ *for* $a = d+1-d/p$, $b = d+1-d/p-N$, *and* $c = d - \delta(p)$,

(ii) $h = (d+1+N)p - (d+1)$ *for* $a = d+1+N-d/p$, $b = d+1-d/p$, *and* $c = d+N-\delta(p)$.

Moreover, if $N > \max\{(d+1)(1/p-1), 1/p\}$, then it easily follows from Lemma 4.1 that

$$|\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : \mathcal{E}_l^a(x,t) > \lambda\}| \lesssim 2^{lh}\,\lambda^{-p},\ \lambda > 0,$$

where $\mathcal{E}_l^a(x,t) = \sum_{j\in\mathbb{N}} \mathcal{E}_{jl}^a(x,t)$.

Proof. It easily follows from polar coordinates, Corollary 2.1, Fubini's theorem, and Chebyshev's inequality. $\square$

5. Weak type estimates on $H^p(\mathbb{R}^{d+1})$, $0<p<1$.

In this section, first of all we shall obtain the uniform weak type estimates of $\mathcal{T}^{\delta(p)}\mathfrak{a}$ on the closed unbounded conical sector in $\mathbb{R}^{d+1}$ when $\mathfrak{a}$ is a (p,N)-atom with $N \geq (d+1)(1/p-1)$. Then we complete its weak type estimates on the whole space $\mathbb{R}^{d+1}$ by applying the translation invariance of the operator $\mathcal{T}^{\delta}$ and H^p spaces.

Proposition 5.1. *Suppose that $\varrho \in C^\infty(\mathbb{R}^d \setminus \{0\})$ is a non-radial homogeneous distance function satisfying $\varrho(t\xi) = t\varrho(\xi)$ whose unit sphere Σ_ϱ is a convex hypersurface of finite type, and let $0<p<1$ be given. If $\mathfrak{a}$ is a (p,N)-atom with $N \geq (d+1)(1/p-1)$ defined on $\mathbb{R}^{d+1}$, then there exists a constant $C = C(d,p)$ such that*

$$\left|\{(x,t)\in\mathbb{R}^{d+1} : |\mathcal{T}^{\delta(p)}\mathfrak{a}(x,t)|\,\chi_{\Gamma_\gamma}(x,t) > \lambda\}\right| \leq C\lambda^{-p},\ \ \lambda>0.$$

Proof. Since $\mathcal{T}^{\delta(p)}$ is translation invariant, we may assume that $\mathfrak{a}$ is supported in a cube Q of diameter $\mathfrak{d}>0$ centered at the origin. We observe that

$$\begin{aligned} &|\{(x,t)\in\mathbb{R}^{d+1} : |\mathcal{T}^{\delta(p)}\mathfrak{a}(x,t)|\,\chi_{\Gamma_\gamma}(x,t) > \lambda\}| \\ &\quad\leq |\{(x,t)\in Q_*\cap\Gamma_\gamma : |\mathcal{T}^{\delta(p)}\mathfrak{a}(x,t)|\,\chi_{\Gamma_\gamma}(x,t) > \lambda/2\}| \\ &\quad+ |\{(x,t)\in Q_*^c\cap\Gamma_\gamma : |\mathcal{T}^{\delta(p)}\mathfrak{a}(x,t)|\,\chi_{\Gamma_\gamma}(x,t) > \lambda/2\}| \end{aligned} \tag{5.1}$$

where Q_* is the cube concentric with Q and with sides of twice the length, and we will show that each term is bounded by $C\,\lambda^{-p}$.

Suppose $(x,t)\in Q_*\cap\Gamma_\gamma$. By Plancherel theorem and Hölder's inequality with $p/2+1/q=1$, we have that

$$\iint_{Q_*\cap\Gamma_\gamma} |\mathcal{T}^{\delta(p)}\mathfrak{a}(x,t)|^p\,dx\,dt \leq C\,||\mathcal{T}^{\delta(p)}\mathfrak{a}||^p_{L^2(\mathbb{R}^{d+1})}\,|Q_*|^{1/q} \leq C.$$

Hence, by Chebyshev's inequality, we have that for all $\lambda>0$,

$$|\{(x,t)\in Q_* : |\mathcal{T}^{\delta(p)}\mathfrak{a}(x,t)|\,\chi_{\Gamma_\gamma}(x,t) > \lambda/2\}| \leq C\,\lambda^{-p}. \tag{5.2}$$

Next we want to estimate the following weak type inequality

$$|\{(x,t)\in Q_*^c : |\mathcal{T}^{\delta(p)}\mathfrak{a}(x,t)|\,\chi_{\Gamma_\gamma}(x,t) > \lambda/2\}| \leq C\,\lambda^{-p},\quad \lambda>0. \tag{5.3}$$

We first assume that $\mathfrak{a}$ is supported in the cube Q^0 of diameter 1 centered at the origin. We consider the case $(x,t)\in(Q^0_*)^c\cap\Gamma_\gamma$. Fix $l>0$. Since $\mathfrak{a}$ is supported in the cube Q^0 of diameter 1, it follows from Lemma 2.3, (3.2), (3.4), Lemma 3.2, Corollary 3.4, (4.1), and (4.2) that

$$\begin{aligned} &|\mathcal{T}_l^{\delta(p)}\mathfrak{a}(x,t)|\,\chi_{(Q^0_*)^c\cap\Gamma_\gamma}(x,t) \\ &\leq 2^{(d+1)l}\iint_{Q^0}|\mathfrak{a}(y,s)|\,|\mathcal{K}_0^{\delta(p)}(2^l(x-y),2^l(t-s))|\,\chi_{(Q^0_*)^c\cap\Gamma_\gamma}(x,t)\,dy\,ds \\ &\leq \iint_{Q^0}\mathcal{A}_l^{d-\delta(p)}(x-y,t-s)\,dy\,ds + \iint_{Q^0}\mathcal{B}_l^{d+1-d/p-N}(x-y,t-s)\,dy\,ds \\ &+ \iint_{Q^0}\mathcal{C}_l^{d+1-d/p}(x-y,t-s)\,dy\,ds + \iint_{Q^0}\mathcal{D}_l^{d+1-d/p-N}(x-y,t-s)\,dy\,ds \\ &+ \iint_{Q^0}\sum_{j\in\mathbb{N}}\mathcal{E}_{jl}^{d+1-d/p}(x-y,t-s)\,dy\,ds \\ &\leq \mathcal{A}_l^{d-\delta(p)}(x,t) + \mathcal{B}_l^{d+1-d/p-N}(x,t) + \mathcal{C}_l^{d+1-d/p}(x,t) + \mathcal{D}_l^{d+1-d/p-N}(x,t) + \sum_{j\in\mathbb{N}}\mathcal{E}_{jl}^{d+1-d/p}(x,t), \end{aligned}$$

where N is a positive integer satisfying $N > \max\{(d+1)(1/p-1), 1/p\}$. Thus, summing up over the indices $j \in \mathbb{N}$ by using Lemma 4.2 and Lemma 4.3, we obtain that

$$\left|\{(x,t) \in (Q_*^0)^c : |\mathcal{T}_l^{\delta(p)}\mathfrak{a}(x,t)|\,\chi_{\Gamma_\gamma}(x,t) > \lambda/4\}\right| \lesssim 2^{(d+1)(p-1)l}\lambda^{-p},\ \lambda > 0.$$

Adding up over the indices $l > 0$ by using Lemma 4.2 once again, we easily get that

$$\left|\{(x,t) \in (Q_*^0)^c : \sum_{l>0} |\mathcal{T}_l^{\delta(p)}\mathfrak{a}(x,t)|\,\chi_{\Gamma_\gamma}(x,t) > \lambda/4\}\right| \lesssim \lambda^{-p},\ \lambda > 0. \tag{5.4}$$

We now fix $l \le 0$. Let $N \in \mathbb{N}$ be an integer satisfying $N-1 \le \max\{(d+1)(1/p-1), 1/p\} < N$. Then we see that $(d+1+N)p-(d+1) > 0$. If $(x,t) \in (Q_*^0)^c \cap \Gamma_\gamma$, let $\mathcal{P}_{l,x,t}(y,s)$ denote the $(N-1)$-th order Taylor polynomial of the function $(y,s) \mapsto \mathcal{K}_l(x-y,t-s)$ expanded near the origin $(0,0) \in \mathbb{R}^d \times \mathbb{R}$. Then we have that $\mathcal{P}_{l,x,t}(x,t) = 2^{(d+1)l}\,\mathcal{P}_{0,x,t}(2^l x, 2^l t)$ for fixed $l \le 0$. Then it follows from the vanishing moment conditions on $\mathfrak{a}$, Lemma 2.3, (3.2), (3.4), Lemma 3.2, Corollary 3.4, (4.1), and (4.2) that

$$\begin{aligned}
&|\mathcal{T}_l^{\delta(p)}\mathfrak{a}(x,t)|\,\chi_{(Q_*^0)^c\cap\Gamma_\gamma}(x,t)\\
&= 2^{(d+1)l}\iint_{Q^0} |\mathfrak{a}(y,s)|\,|\mathcal{K}_0^{\delta(p)}(2^l(x-y),2^l(t-s)) - \mathcal{P}_{0,x,t}(2^l y, 2^l s)|\,\chi_{(Q_*^0)^c\cap\Gamma_\gamma}(x,t)\,dy\,ds\\
&\lesssim 2^{(d+1)l}\int_0^1\iint_{Q^0}\sum_{|\alpha|=N}\frac{1}{\alpha!}\,|[\mathcal{D}^\alpha\mathcal{K}_0^{\delta(p)}](2^l(x-\tau y),2^l(t-\tau s))|\,|2^l(y,s)|^N\,\chi_{(Q_*^0)^c\cap\Gamma_\gamma}(x,t)\,dy\,ds\,d\tau\\
&\le \int_0^1\iint_{Q^0}\mathcal{A}_l^{d+N-\delta(p)}(x-\tau y,t-\tau s)\,dy\,ds\,d\tau + \int_0^1\iint_{Q^0}\mathcal{B}_l^{d+1-d/p}(x-\tau y,t-\tau s)\,dy\,ds\,d\tau\\
&+ \int_0^1\iint_{Q^0}\mathcal{C}_l^{d+1+N-d/p}(x-\tau y,t-\tau s)\,dy\,ds\,d\tau + \int_0^1\iint_{Q^0}\mathcal{D}_l^{d+1-d/p}(x-\tau y,t-\tau s)\,dy\,ds\,d\tau\\
&+ \int_0^1\iint_{Q^0}\sum_{j\in\mathbb{N}}\mathcal{E}_{jl}^{d+1+N-d/p}(x-\tau y,t-\tau s)\,dy\,ds\,d\tau\\
&\le \mathcal{A}_l^{d+N-\delta(p)}(x,t) + \mathcal{B}_l^{d+1-d/p}(x,t) + \mathcal{C}_l^{d+1+N-d/p}(x,t) + \mathcal{D}_l^{d+1-d/p}(x,t) + \sum_{j\in\mathbb{N}}\mathcal{E}_{jl}^{d+1+N-d/p}(x,t).
\end{aligned}$$

Therefore summing up over the indices $j \in \mathbb{N}$ by using Lemma 4.2 and Lemma 4.3 leads us to obtain the following weak type estimate

$$\left|\{(x,t) \in (Q_*^0)^c : |\mathcal{T}_l^{\delta(p)}\mathfrak{a}(x,t)|\,\chi_{\Gamma_\gamma}(x,t) > \lambda/4\}\right| \lesssim 2^{[(d+1+N)p-(d+1)]l}\lambda^{-p},\ \lambda > 0.$$

Adding up over the indices $l \le 0$ by using Lemma 4.2 once again, we easily obtain that

$$\left|\{(x,t) \in (Q_*^0)^c : \sum_{l\le 0} |\mathcal{T}_l^{\delta(p)}\mathfrak{a}(x,t)|\,\chi_{\Gamma_\gamma}(x,t) > \lambda/4\}\right| \lesssim \lambda^{-p},\ \lambda > 0. \tag{5.5}$$

Suppose now that $\mathfrak{a}$ is an arbitrary (p,N)-atom ($N \ge (d+1)(1/p-1)$) supported in a cube Q of diameter $\mathfrak{d} > 0$ centered at $(x_0,t_0) \in \mathbb{R}^{d+1}$. Let $\mathfrak{b}(x,t) = \mathfrak{d}^{(d+1)/p}\,\mathfrak{a}(\mathfrak{d}\,(x-x_0), \mathfrak{d}\,(t-t_0))$. Since $\mathcal{T}^{\delta(p)}$ is translation invariant, without loss of generality we may assume that $(x_0,t_0) = (0,0)$. Then

$\mathfrak{b}$ is an atom supported in the cube Q^0 of diameter 1 centered at the origin $(0,0) \in \mathbb{R}^d \times \mathbb{R}$. This implies that

$$\mathcal{T}_l^{\delta(p)} \mathfrak{a}(x,t) = \mathfrak{d}^{-(d+1)/p} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathfrak{b}\left(\frac{x-y}{\mathfrak{d}}, \frac{t-s}{\mathfrak{d}}\right) \mathcal{K}_l^{\delta(p)}(y,s)\, dy\, ds$$
$$= \mathfrak{d}^{-(d+1)(1/p-1)}\, \mathfrak{b} * \mathcal{K}_l^{\delta(p)}(\mathfrak{d}\cdot, \mathfrak{d}\cdot)\left(\frac{x}{\mathfrak{d}}, \frac{t}{\mathfrak{d}}\right). \tag{5.6}$$

Repeating the same arguments used in (5.4) and (5.5) in terms of (5.6), we obtain the weak type estimate (5.3) given in the above. Hence by (5.1), (5.2), and (5.3) we complete the proof. □

Proof of Theorem 1.1. Let $f = \sum_{i=1}^{\infty} c_i \mathfrak{a}_i \in H^p(\mathbb{R}^{d+1})$ where $\mathfrak{a}_i's$ are (p,N)-atom $(N \geq (d+1)(1/p-1))$. Then we see that

$$\|f\|_{H^p} \sim \sum_{i=1}^{\infty} |c_i|^p < \infty.$$

By Proposition 5.1, we obtain that

$$\left|\{(x,t) \in \Gamma_\gamma : |\mathcal{T}^{\delta(p)} \mathfrak{a}_i(x,t)| > \lambda\}\right| \lesssim \lambda^{-p},$$

where the constant C does not depend upon λ and $\mathfrak{a}_i$. Thus by applying Stein, Taibleson, and Weiss's lemma (see [10]), we have that

$$\left|\{(x,t) \in \Gamma_\gamma : |\mathcal{T}^{\delta(p)} f(x,t)| > \lambda\}\right| \lesssim \lambda^{-p} \sum_{i=1}^{\infty} |c_i|^p. \tag{5.7}$$

Finally, it remains to show that the inequality (5.7) holds on $\mathbb{R}^{d+1}$. For any $R > 0$ there must be a ball $B_R(x_0,t_0)$ of radius R centered at a point (x_0,t_0) contained in the conical sector Γ_γ. Then it follows that

$$\left|\{(x,t) \in B_R(x_0,t_0) : |\mathcal{T}^{\delta(p)} f(x,t)| > \lambda\}\right| \leq \left|\{(x,t) \in \Gamma_\gamma : |\mathcal{T}^{\delta(p)} f(x,t)| > \lambda\}\right|. \tag{5.8}$$

Now we define $(\tau_{hk} f)(x,y) = f(x-h, y-k)$, $h \in \mathbb{R}^d, k \in \mathbb{R}$. Since the operator $\mathcal{T}^{\delta(p)}$ is translation invariant and commutes with the translation operator (i.e. $\tau_{hk}(\mathcal{T}^{\delta(p)} f) = \mathcal{T}^{\delta(p)}(\tau_{hk} f)$), the left-hand side of (5.8) is rewritten as

$$\begin{aligned}
&\left|\{(x,t) \in B_R(x_0,t_0) : |\mathcal{T}^{\delta(p)} f(x,t)| > \lambda\}\right| \\
&\quad = \left|\{(x',t') \in B_R(0,0) : |\tau_{(-x_0)(-y_0)}(\mathcal{T}^{\delta(p)} f)(x',t')| > \lambda\}\right| \\
&\quad = \left|\{(x',t') \in B_R(0,0) : |[\mathcal{T}^{\delta(p)}(\tau_{(-x_0)(-y_0)} f)](x',t')| > \lambda\}\right|.
\end{aligned}$$

Thus it follows from (5.8) and the fact $||(\tau_{(-x_0)(-y_0)} f)||_{H^p} = ||f||_{H^p}$ that

$$\left|\{(x,t) \in B_R(0,0) : |\mathcal{T}^{\delta(p)} f(x,t)| > \lambda\}\right| \lesssim \lambda^{-p} \sum_{i=1}^{\infty} |c_i|^p. \tag{5.9}$$

Therefore, the inequality (5.9) being uniform in $R > 0$ implies that

$$\left|\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : |\mathcal{T}^{\delta(p)} f(x,t)| > \lambda\}\right| \lesssim \lambda^{-p} \sum_{i=1}^{\infty} |c_i|^p \lesssim \frac{\|f\|_{H^p}}{\lambda^p}.$$

Hence we complete the proof. □

Remark. If $\delta < d(1/p-1/2) - 1/2$, it is easily shown that $\mathcal{T}^\delta$ is not weak type (p,p) on $H^p(\mathbb{R}^{d+1})$ from Lemma 4.3. Moreover, the inequalities in Lemma 4.3 are sharp (see [10], p.90).

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